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A CANONICAL FORM FOR NONLINEAR SYSTEMS

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Abstract

The conceptions of transformation and canonical form have been much used to analyze the structure of linear systems. In this article we extend the ideas to nonlinear systems. A coordinate system and a corresponding canonical form are developed for general nonlinear control systems. Their usefulness is demonstrated by showing that every feedback linearizable system becomes a system with only feedback paths in the canonical form.

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I. Introduction.

In a setting of great generality, we say that two control systems

$$\dot{x} = f(x,u) \tag{1}$$

and

$$\dot{y} = g(y,v) \tag{2}$$

are <u>feedback equivalent</u> if there exist two mappings

$$T: \mathbb{R}^n \to \mathbb{R}^n \text{ (invertible)}$$

and

$$W: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \text{ (invertible with respect to the second variable)}$$

such that for any $(x(t),u(t))$ satisfying equation (1), the induced pair of time functions $(T(x(t)), W(x(t),u(t)))$ satisfies equation (2), with substitution of $y(t)$ by $T(t)$ and $v(t)$ by $W(t)$. Recently a fair amount of attention has been paid to the study of a special equivalence class in this sense called the linear equivalents. It is now known [1] that a nonlinear system

$$\dot{x} = f(x) + g(x)\, u \tag{3}$$

of n states and one input is (locally) feedback equivalent to a controllable linear system of the same dimension

$$\dot{y} = Ay + bv \tag{4}$$

if and only if both conditions

(i) the vector fields $g, ad^1 f(g),\ldots,ad^{n-1} f(g)$ are linearly independent, and

(ii) the vector fields $g, ad^1 f(g),\ldots,ad^{n-2} f(g)$ are involutive,

are satisfied. Such a system (3) is called a (feedback) linearizable system. The symbols $ad^k f(g)$ denote the Lie brackets, namely, the vector fields defined by

$$ad^0 f(g) = g,\ ad^1 f(g) = [f,g] = \frac{\partial f}{\partial x}g - \frac{\partial g}{\partial x}f,\ ad^k f(g) = [f,\ ad^{k-1} f(g)].$$

A set of vector fields $X_1, X_2, \ldots X_m$ are said to be <u>involutive</u> if there exist scalars α_{ijk} such that

$$[X_i, X_j] = \sum_{k=1}^{m} \alpha_{ijk} X_k \quad , 1 \leq i,j \leq m.$$

When a system satisfies conditions (i) and (ii), a set of partial differential equations

$$\begin{aligned} &< dT_1,\ ad^k f(g) > = 0\ , \quad k = o,1,\ldots,n-2, \\ &< dT_1,\ ad^{n-1} f(g) > \neq 0, \end{aligned} \tag{5}$$

which defines the leading variable T_1 of linearizing transformations, is then solvable; with a particular solution T_1 the rest of a transformation can be constructed by defining

$$\begin{aligned} T_{i+1} &= < dT_i, f > \quad , \quad i = 1,2,\ldots,n-1, \\ W &= < dT_n,\ f + gu> \ . \end{aligned} \tag{6}$$

In order to solve this overdetermined system of partial differential equations (5), in a recent paper [2] we constructed a state space coordinate change through a set of ordinary differential equations, such that one of the new states, as a function of the original states, can serve as the leading variable of a transformation.

In this article we would like to put the foregoing coordinate change in a broader perspective. The same scheme for the coordinate change will be followed, but in the context of general (not necessarily linearizable) systems. As a result, we find:

(a) The resulting coordinate system brings a nonlinear system into a canonical form, which shows a separation of the linearizable dynamics and the nonlinearizable.

(b) Every linearizable system in this canonical form becomes a system with only feedback paths.

II. Canonical Form

The systems to be considered are of the form

$$\dot{x} = f(x) + g(x)\,u \tag{7}$$

with the vector fields f and g being analytic on a neighborhood $U \subset \mathbb{R}^n$ containing the origin. It will always be assumed that the vector fields g, $ad^1 f(g), \ldots,$ and $ad^{n-1} f(g)$ associated with the system are linearly independent on U.

For any analytic vector field X on U, the <u>one-parameter group generated by X</u> is an analytic mapping $\phi_X : I \times U \to \mathbb{R}^n$, where I is an open interval of $\mathbb{R}$ containing the origin, such that

$$\frac{d}{dt}\,\phi_X(t,p) = X(\phi_X(t,p))$$

and

$$\phi_X(o,p) = p, \quad t \in I,\ p \in U.$$

Restricted to a single point p in U, the mapping $\phi_X(\cdot,p)$ defines an integral curve of X in U, with p being its initial condition. In general, if restricted to a k-dimensional manifold $\mathcal{S}$, which is assumed to be transversal to the integral curves of X, then the mapping $\phi_X(\cdot,p)$, $p \in \mathcal{S}$, defines a (k+1) - dimensional manifold, to every point q in which the vector X(q) is tangent.

Let ϕ_k be the one-parameter group generated by $ad^k f(g)$ for $k = 0,1,\ldots,n-1$. We assume, without loss of generality, that $I \times \mathcal{U}$ be the common domain for all ϕ_k. With the integral curves of $ad^k f(g)$, we now construct a sequence of manifolds:

$$S_o = \{0\} \ ,$$

$$S_k = \{\phi_{n-k}(t,p) \mid t \in I,\ p \in S_{i-1}\} \cap \mathcal{U} \tag{8}$$

for $k = 1,2,\ldots,n$. Thus, S_1 is a one-dimensional manifold made of an integral curve of $ad^{n-1}f(g)$ with the origin being the initial condition. And for each k, S_k is a k-dimensional manifold resulting from the union of the integral curves of $ad^{n-k}f(g)$ with S_{k-1} being the initial condition. Consequently, we have

$$\{0\} \subset S_1 \subset S_2 \subset \ldots \subset S_n \subset \mathcal{U}.$$

These manifolds S_k are precisely what we would obtain when we follow the integration procedure prescribed in [2] for finding the leading variable (the T_1 in (5)) of a linearizing transformation for a linearizable system. We remark, however, that the involutiveness condition, and hence linearizability is not assumed here.

We are now in a position to define a particular coordinate system for equation (7) in the neighborhood S_n. For any point $p \in S_n$, there is a unique integral curve of g connecting p and S_{n-1}, intersecting S_{n-1} at a single point, say, q. It is clear that q and p are related by the one-parameter group generated by g, that is, $\phi_o(s_1,q) = p$, for some uniquely determined parameter s_1. In like manner, the point q is connected by an integral curve of $ad^1 f(g)$ with S_{n-2} at a point $r \in S_{n-2}$. Again, a unique parameter s_2 is determined such that $\phi_1(s_2,r) = q$. Continuing this protill we finally reach the origin by an integral curve of $ad^{n-1}f(g)$, we should obtain a set of n parameters $(s_1, s_2, \ldots, s_n)$, which is uniquely associated with the starting point p in S_n, with the result that

$$\phi_0\ (s_1, (\phi_1(s_2,\ (\ldots(\phi_{n-1}(s_n, 0)\ldots) = p. \tag{9}$$

The mapping from $(s_1, s_2, \ldots, s_n)$ to p is clearly analytic. And from the standing assumption of linearly independency on g, $ad^1 f(g), \ldots, ad^{n-1}f(g)$, the mapping is also nonsingular. Therefore, we have established an analytic coordinate change from an arbitrary coordinate system x to the particular coordinate system s, denoted by $F : x \mapsto s$. Under the coordinate change $F : x \mapsto s$ a system equation (7) becomes

$$\dot{s} = \frac{\partial F}{\partial x} f(F^{-1}(s)) + \frac{\partial F}{\partial x} g(F^{-1}(s)) \cdot u, \tag{10}$$

which, for notational convenience, will still be expressed as

$$\dot{s} = f(s) + g(s)\ u \tag{11}$$

The coordinate system s is defined on the intrinsic geometry of a control system. Naturally, it is closely related to other geometric characters of a system, such as feedback linearizability. To see this, we shall examine a canonical form of nonlinear systems resulting from the coordinates s. Before doing so, we need to make two observations.

First, in the coordinate system s, the manifolds $S_k, k = 1, 2, \ldots, n-1$, are "flat" manifolds, or more precisely

$$S_k = \{s \in \mathbb{R}^n \mid s_i = 0;\ 1 \leq i \leq n-k\}\ .$$

This can easily be deduced from the integration procedure and the definition of the coordinates $s_1, s_2, \ldots, s_n$.

Secondly, because the coordinate s_1 of every point $p \in S_n$ is defined by the parameter of the one-parameter group of g, that is

$$\phi_0(s_1, q) = p,$$

where q is in S_{n-1} which is transversal with the integral curves of g, the vector field g becomes exactly

$$g = \frac{\partial}{\partial s_1} ,$$

or

$$g = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{12}$$

This is an elementary fact in differential geometry; it can be verified by simple computation. For detail, the reader is referred to, for instance, Spivak [3]. By repeatedly using this argument, we conclude that the vector field $ad^k f(g)$ is

$$ad^k f(g) = \frac{\partial}{\partial s_{k+1}}$$

or

$$ad^k f(g) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow (k+1)\text{-th place}, \tag{13}$$

when it is restricted to the manifold S_{n-k}, $k = 0,1,\ldots, n-1$.

In the following we analyze equation (11) and construct a canonical form, using the foregoing observations. Equation (11) can now be written as

$$\dot{s} = f(s) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \tag{14}$$

By Taylor series expansion the vector field f(s) can be expanded with respect to s_1 as

$$f(s) = f(0,s_2,\ldots,s_n) + \sum_{k=1}^{\infty} \frac{s_1^k}{k!} \frac{\partial^k f}{\partial s_1^k} (0,s_2,\ldots,s_n) \tag{15}$$

We already know that $f(0,s_2,\ldots,s_n)$ and $\frac{\partial^k f}{\partial s_1^k}(0,s_2,\ldots,s_n)$ are the corresponding vector fields restricted to the manifold S_{n-1}. Moreover, since $g = \frac{\partial}{\partial s_1}$ in S_n, we have

$$\frac{\partial^k f}{\partial s_1^k} = (-1)^k \; ad^k g(f) \;,$$

and when $k = 1$,

$$\begin{aligned}\frac{\partial f}{\partial s_1}(0,s_2,\ldots s_n) &= - ad^1 g(f) \\ &= ad^1 f(g) \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .\end{aligned}$$

Therefore the vector field f(s) can actually be expressed as

$$f(s) = f(0,s_2,\ldots,s_n) + s_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{k=2}^{\infty} (-1)^k \frac{s_1^k}{k!} ad^k g(f) \; (0,s_2,\ldots,s_n) \quad (16)$$

The same analysis can again be performed on the first term, $f(0,s_2,\ldots,s_n)$, the vector field f is restricted to S_{n-1}. Since the notation becomes too cumbersome, we denote the vector fields $ad^k f(g)$ by X_k to keep the expressions compact.

Again, by Taylor series expansion, this time with respect to s_2, we have

$$f(0,s_2,\ldots,s_n) = F(0,0,s_3,\ldots,s_n) + \sum_{k=1}^{\infty} \frac{s_2^k}{k!} \frac{\partial^k f}{\partial s_2^k} \; (0,0,s_3,\ldots,s_n) \qquad (17)$$

We recognize that $\frac{\partial^k f}{\partial s_2^k}(0,0,s_3,\ldots,s_n) = (-1)^k \; ad^k_{X_1}(f) \,|\, _{S_{n-2}}$

$ad_{X_1}{}^k(f)$

and when k=1,

$$-ad^1 X_1(g) = ad^1 f(X_1)$$

$$= ad^2 f(g)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} .$$

So equation (16) becomes

$$f(0,s_2,\ldots,s_n) = f(0,0,s_3,\ldots s_n) + s_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$+ \sum_{k=2}^{\infty} (-1)^k \frac{s_2^k}{k!} \cancel{\frac{\partial^k f}{\partial s_2^k}}\, ad^k X_1(f)\ (0,0,s_3,\ldots,s_n)$$

We continue the expansion and analysis in like manner. At the ℓ-th step we have

$$f(0,\ldots,0,s_\ell,s_{\ell+1},\ldots s_n)$$

$$= f(0,\ldots,0,0,s_{\ell+1},\ldots,s_n) + s_\ell \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$+ \sum_{k=2}^{\infty} (-1)^k \frac{s_\ell^k}{k!} \cancel{\frac{\partial^k f}{\partial s_\ell^k}}\, ad^k X_{\ell-1}(f)\ (0,\ldots,0,s_{\ell+1},\ldots,s_n),$$

where, in the second term, 1 occurs at the $(\ell+1)$-th place, $\ell=1,\ldots,n-1$. At the n-th step we expand $f(0,\ldots,0,s_n)$ as

$$f(0,\ldots,0,s_n) = f(0) + \sum_{k=1}^{\infty} (-1)^k \frac{s_n^k}{k!} ad^k X_{n-1}(f)(0).$$

Here, however, no special form can be concluded for any term in the summation. In summary, we have established a canonical form for the system equation (11) as follows:

$$\dot{s} = u\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + s_1\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + \sum_{k=2}^{\infty}(-1)^k \frac{s_1^k}{k!}\,ad^k X_0(f)|_{S_{n-1}}$$

$$+ \dots$$

$$+ s_{n-1}\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix} + \sum_{k=2}^{\infty}(-1)^k \frac{s_{n-1}^k}{k!}\,ad^k X_{n-2}(f)|_{S_1}$$

$$+ \sum_{k=1}^{\infty}(-1)^k \frac{s_n^k}{k!}\,ad^k X_{n-1}(f)(0) + f(0), \qquad (19)$$

Where the symbol $|_{S_j}$ attached to a vector field means the vector field is evaluated at $(0,\dots 0,s_{n-j+1},\dots,s_n)$.

Equation (19) can be rearranged into the following expression

$$\dot{s} = f(0) + \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} u + \begin{pmatrix}0 & 0 & & 0 & *\\ 1 & 0 & & 0 & *\\ 0 & 1 & \dots & 0 & *\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & & 1 & *\end{pmatrix}\begin{pmatrix}s_1\\s_2\\s_3\\\vdots\\s_n\end{pmatrix} \qquad (20)$$

$$+ \sum_{k=2}^{\infty}\frac{1}{k!}\left(ad^k X_0(f)|_{S_{n-1}}, \dots, ad^k X_{n-1}(f)|_0\right)\begin{pmatrix}s_1^k\\\vdots\\s_n^k\end{pmatrix}$$

where the last column $\begin{pmatrix}*\\*\\\vdots\\*\end{pmatrix}$ in the third term is $ad^1 X_{n-1}(f)|_0$, which has no special form. Equation (20) is taken to be the canonical form for a nonlinear system (11) in the s coordinates. It is a very convenient representation for geometric analysis of control systems. In the following we shall discuss one of its applications in feedback linearizability.

IV. Linearizability and Pure Feedback Systems.

This section is devoted to the question: How much more can we say about the canonical form when a system is feedback linearizable? The answer turns out to be quite pleasing.

Let us consider a linear system

$$\dot{x} = Ax + bu \tag{21}$$

of n states and one input. With Ax identified with f(x) and b with g(x), the vector fields $ad^k f(g)$ are exactly the familiar vectors $A^k b$. Being constant vector fields in $\mathbb{R}^n$, their integral curves are straight lines, and the defining equation for them,

$$\phi_{X_k}(t,q) = p,$$

becomes

$$t\, A^k b + q = p.$$

This observation brings equations (9), which defines the coordinates s, into

$$s_1 b + s_2 Ab + \ldots + s_n A^{n-1} b = x \tag{22}$$

in the linear case, where x is some arbitraty coordinate system. Equation (22) defines a linear transformation from x-space to s-space, which is more apparent when it is rewritten as

$$[b, Ab, \ldots, A^{n-1}b]\, s = x$$

or

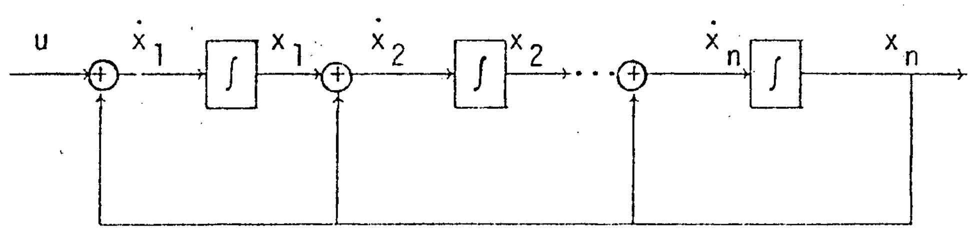
$$x = [b, Ab, \ldots, A^{n-1}b]^{-1} x.$$

This is a standard transformation in linear theory to change a linear system into the canonical form

$$\dot{s} = \begin{pmatrix} 0 & 0 & & 0 & * \\ 1 & 0 & & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & * \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u. \tag{24}$$

This result can easily be verified by direct computation within the linear framework. It can also be seen from specialization of the nonlinear canonical form (20), using the fact that the origin is an equilibrium state and the second and higher order Lie brackets are vanishing.

The linear canonical form (24) is of course already well-known. However, we want to emphasize a special feature of its structure, which is crucial to the following development. We first represent the form (24) by block diagram.



From this block diagram we note that there are only feedback paths, and the main forward path, but no feedforward paths. Such a system is to be called a <u>pure feedback system</u>. More formally, a nonlinear system (7) is called a <u>pure feedback system</u> if it is of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,\dots x_n) + g_1(x_1,\dots,x_n)\, u \\ \dot{x}_2 &= f_2(x_1,\dots,x_n) \\ &\ \vdots \\ \dot{x}_k &= f_k(x_{k-1},\dots x_n) \\ &\ \vdots \\ \dot{x}_n &= f_n(x_{n-1}\ ,\ x_n)\ . \end{aligned} \tag{25}$$

We leave it to the reader to check that this system has no feedforward paths in its block diagram.

Pure feedback systems are very desirable in many ways. First, the involutiveness condition for linearizability is trivially satisfied. In (24)

$$g = \begin{pmatrix} g_1(x_1,\dots,x_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ f_2(x_1,\dots,x_n) \\ \vdots \\ f_n(x_{n-1},x_n) \end{pmatrix}.$$

Computing, we have

$$[f,g] = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \dots & & \frac{\partial f_2}{\partial x_n} \\ 0 & \frac{\partial f_3}{\partial x_2} & \dots & \frac{\partial f_3}{\partial x_n} \\ & \dots & & \\ 0\dots 0 & & \frac{f_n}{x_{n-1}} & \frac{f_n}{x_n} \end{pmatrix} \begin{pmatrix} g_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ 0 & \dots & 0 \\ & \dots & \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} g_1 - \sum_{i=1}^{n} \frac{\partial g_1}{\partial x_i} f_i \\ \frac{\partial f_2}{\partial x_1} g_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$[g,[f,g]] = \begin{pmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where asterisks denote possibly nonzero elements. From the form of $[g[f,g]]$, it is clearly a linear combination of g and $[f,g]$, and hence, the vector fields g and $[f,g]$ are involutive. In a similar manner one can verify that the vector fields g, $ad^1 f(g),\ldots,ad^k f(g)$ are indeed involutive for $k = 0,1,\ldots,n-2$.

For pure feedback systems, the condition that the vector fields g, $ad^1 f(g),\ldots,ad^{n-1} f(g)$ are linearly independent is also easy to check. One can show that this condition is exactly equivalent to that

$$g_1, \frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_k}{\partial x_{k-1}}, \ldots, \frac{\partial f_n}{\partial x_{n-1}} \tag{26}$$

being non-vanishing.

The proof is straightforward and is omitted here.

Another nice feature of a pure feedback system (24) is the ease with which a linearizing transformation is constructed. We can simply choose x_n to be the leading canonical variable; we denote it by y_n. Let $f_n(x_{n-1},x_n)$ be the second canonical variable y_{n-1}, then

$$\dot{y}_n = y_{n-1}.$$

Next, define

$$\begin{aligned} y_{n-2} = \dot{y}_{n-1} &= \frac{\partial f_n}{\partial x_{n-1}} \dot{x}_{n-1} + \frac{\partial f_n}{\partial x_n} \dot{x}_n \\ &= \frac{\partial f_n}{\partial x_{n-1}} f_{n-1} + \frac{\partial f_n}{\partial x_n} f_n , \end{aligned}$$

so we have $\dot{y}_{n-1} = y_{n-2}$. We continue this differentiation process until we reach $\dot{y}_1$, which is a function of the form $F_1(x) + F_2(x)u$, and we define it to be the new control variable v, implying $\dot{y}_1 = v$. In summation, we have

$$
\begin{aligned}
\dot{y}_1 &= v \\
\dot{y}_2 &= y_1 \\
&\cdots \\
\dot{y}_n &= y_{n-1} \, .
\end{aligned}
\tag{27}
$$

When the condition (26) is satisfied, the transformation: $x \mapsto y$, $(x,u) \mapsto v$, is indeed a legitimate feedback transformation, which brings a pure feedback system (24) into "a series of integrators" (27).

The class of pure feedback systems was first recognized by George Meyer who with his colleagues at NASA Ames Research Center incorporated the ideas of transformation and canonical form in nonlinear control system designs. For their design results, the reader is referred to the papers [4,5].

We now come to an application of our canonical form (20).

Theorem 1. Every feedback linearizable system is a pure feedback system.

Proof. Recall the canonical form of a nonlinear system in the coordinate system s

$$
s = f(0) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 & & 0 & * \\ 1 & 0 & & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & & 1 & * \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{pmatrix}
$$

$$
+ \sum_{k=2}^{\infty} \frac{1}{k!} \left(ad^k X_0(f) \quad S_{n-1} \quad \cdots \quad ad^k X_{n-1}(f)|_0 \right) \begin{pmatrix} s_1^k \\ \vdots \\ s_n^k \end{pmatrix} , \tag{28}
$$

where $X_i = ad^i f(g)$. Suppose it is linearizable, which implies that the vector fields $X_0, X_1, \ldots, X_{n-2}$ are involutive.

It has been shown in [1] that the involutiveness of $X_0, X_1, \ldots X_{n-2}$ implies that the sets $\{X_0, X_1, \ldots X_i\}$ are involutive for all $0 \leq i \leq n-2$. It is also easy to varify that if two vector fields X and Y are involutive, then all the higher order Lie brackets of X and Y are expressible as linear combinations of X and Y.

Consider now the first column of the matrix in the summation in (28). From the remarks just made and the fact that

$$X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

on S_{n-1}, there exist two scalar functions $\alpha_{1,k}$ and $\alpha_{2,k}$ such that

$$\begin{aligned} ad^k X_0(f)\Big|_{S_{n-1}} &= \alpha_{1,k}(s)\, X_0\Big|_{S_{n-1}} + \alpha_{2,k}(s) X_1\Big|_{S_{n-1}} \\ &= \alpha_{1,k}(0,s_2,\ldots s_n)\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_{2,k}(0,s_2,\ldots,s_n)\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{1,k}(0,s_2,\ldots,s_n) \\ \alpha_{2,k}(0,s_2,\ldots,s_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

for $k \geq 2$.

In a similar manner we can conclude that linearizability implies

$$
\mathrm{ad}^k X_\ell(f)\Big|_{S_{n-\ell-1}} = \begin{pmatrix} *(0,\dots,0,s_{\ell+1},\dots,s_n) \\ \vdots \\ *(0,\dots,0,s_{\ell+1},\dots,s_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow(\ell+2)\text{-th place} \\ \\ \\ \\ \end{matrix}
$$

for $k \geq 2$. In summation, equation (27) becomes

$$
\dot{s} = f(0) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 & & 0 & * \\ 1 & 0 & & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & * \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{pmatrix}
$$

$$
+ \sum_{k=2}^{\infty} \frac{1}{k!} \begin{pmatrix} *(s_2,\dots,s_n) & *(s_3,\dots,s_n) & & *(s_n) & * \\ *(s_2,\dots,s_n) & *(s_3,\dots,s_n) & \dots & *(s_n) & * \\ & *(s_3,\dots,s_n) & & *(s_n) & * \\ & & \ddots & \vdots & \\ & & & *(s_n) & * \end{pmatrix} \begin{pmatrix} s_1^k \\ s_2^k \\ s_3^k \\ \vdots \\ s_n^k \end{pmatrix}, \quad (29)
$$

where $*(s_j,\dots,s_n)$ are some scalar unspecified functions and the last column $*$'s are constants. Because of the analyticity assumed, all the infinite series are summable. Thus we have

$$
\dot{s} = f(0) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 & & 0 & * \\ 1 & 0 & & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & * \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{pmatrix}
$$

$$
+ \begin{pmatrix} *(s_1,\ldots,s_n) \\ *(s_1,\ldots,s_n) \\ *(s_2,\ldots,s_n) \\ \vdots \\ *(s_{\ell-1},\ldots,s_n) \\ \vdots \\ *(s_{n-1},s_n) \end{pmatrix} \leftarrow \ell\text{th place.}
$$

This is clearly a pure feedback system, and the proof is completed. □

IV. Conclusion.

The conceptions of transformation and canonical form have been much used to analyze the structure of linear systems. In this article we have extended the ideas to nonlinear systems.

The proposed coordinate system and canonical form are established on the intrinsic geometric nature of the system. They are especially useful in analyzing geometric properties of a system such as feedback linearizability. Although only discussed in the context of single-input systems, a multi-input version of the theory can be constructed in essentially the same manner, which, as well as its computational aspects, are being investigated by the authors.

Our practical motivation of this study lies in robustness analysis for the design methodology developed by George Meyer and his colleagues at NASA Ames Research Center [5], which depends crucially on feedback linearizability of the nonlinear plant to be controlled. The canonical form described in this article offers a model of the plant on which the designer can make a best model reduction for feedback linearization purposes. We also view the form as a starting point from where a theory of robustness of the design can possibly be constructed.

V. References.

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